Quantum cloning, Bell's inequality and teleportation

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Abstract
We analyse the possibility of using the two-qubit output state from the Buzek–Hillery quantum copying machine (not necessarily a universal quantum cloning machine) as a teleportation channel. We show that there is a range of values of the machine parameter ξ for which the two-qubit output state is entangled and violates the Bell-CHSH inequality and for a different range it remains entangled but does not violate the Bell-CHSH inequality. Further, we observe that for certain values of the machine parameter the two-qubit mixed state can be used as a teleportation channel. The use of the output state from the Buzek–Hillery cloning machine as a teleportation channel provides an additional appeal to the cloning machine and motivation for our present work.

PACS number: 03.67.−a

1. Introduction

In 1993, an international team of six scientists including Charles Bennett [4] discovered a new aspect of quantum inseparability—teleportation. Teleportation is purely based on classical information and non-classical Einstein–Podolsky–Rosen(EPR) correlations. The basic idea is to use a pair of particles in a singlet state shared by distant partners Alice and Bob to perfom successful teleportation of an arbitrary qubit from the sender Alice to the receiver Bob. There was a question regarding what value of fidelity of transmission of an unknown state can assure us about the non-classical character of the state forming the quantum channel. It has been shown that the purely classical channel can give at most $F = \frac{2}{3}$ [3, 5, 10].

Then Popescu raised basic questions concerning a possible relation between teleportation, the Bell-CHSH inequalities [6] and inseparability: 'what is the exact relation between Bell's inequalities violation and teleportation?'[3]. In this paper, we probe these relations for the two-qubit mixed entangled state arising from the B–H quantum cloning machine.

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Bell's inequalities are relations between conditional probabilities valid under the locality assumption. Hence, *a priori* they have nothing to do with quantum mechanics. However, it is the fact that quantum mechanics predicts a violation of these conditions which makes them interesting. Werner [9] gave an example of an entangled state described by the density operator $\rho_W = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}I$, where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and I is the identity operator in the four-dimensional Hilbert space, which does not violate the Bell inequality for $\frac{1}{3} < p < \frac{1}{\sqrt{2}}$. Interestingly in this work, we also find an example of an entangled state that does not violate Bell's inequality. A natural question arises concerning teleportation, whether states which violate the Bell-CHSH inequalities are suitable for teleportation. Horodecki *et al* [2] showed that any mixed two spin-$\frac{1}{2}$ state which violates the Bell-CHSH inequalities is suitable for teleportation.

With the advent of a quantum cloning machine we were introduced to an entanglement between the output and input states. The proposition of a universal quantum copying machine by Buzek and Hillery [1] presented us with a cloning machine which is independent of the input state. However, as was quite expected the copy and original which appear at the output remained entangled.

Our investigation starts from the idea of using the two-qubit entangled state which comes as an output of the B–H cloning machine as a teleportation channel. Central to this idea is the possible utility of a mixed entangled state as a teleportation channel, which motivates us for this work. Our analysis results in a certain range of the machine parameter ξ for which the entangled state can be used as a teleportation channel. For $\xi = \frac{1}{2}$, the output state reduces to a maximally entangled pure state that can be used as a teleportation channel faithfully as expected. Altogether this work provides an additional appeal for the B–H cloning machine.

Our work is organized as follows: in the following section, we discuss the viability of using the output state given by the B–H cloning machine as a teleportation channel. Finally, we summarize our work.

2. Analysis of the output of the Buzek-Hillery copying machine

In this section, we study the viability of the entangled output copies of the Buzek–Hillery cloning machine [1] as a teleportation channel.

The cloning transformation for the copying procedure [1] is given by

$$|0\rangle_a|0\rangle_b|Q\rangle_x \longrightarrow |0\rangle_a|0\rangle_b|Q_0\rangle_x + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b]|Y_0\rangle_x \tag{1}$$

$$|1\rangle_a|0\rangle_b|Q\rangle_x \longrightarrow |1\rangle_a|1\rangle_b|Q_1\rangle_x + [|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b]|Y_1\rangle_x. \tag{2}$$

The unitarity and the orthogonality of the cloning transformation give

$$_x\langle Q_i|Q_i\rangle_x + 2_x\langle Y_i|Y_i\rangle_x = 1 \quad (i = 0, 1) \tag{3}$$

$$_x\langle Y_0|Y_1\rangle_x = {}_x\langle Y_1|Y_0\rangle_x = 0. \tag{4}$$

Here the copying machine state vectors $|Y_i\rangle_x$ and $|Q_i\rangle_x$ are assumed to be mutually orthogonal, so are the state vectors $\{|Q_0\rangle, |Q_1\rangle\}$.

Let us consider a quantum state which is to be cloned,

$$|\chi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{5}$$

where $\alpha^2 + \beta^2 = 1$.

Here we confine ourselves to a limited class of input states (5), where α and β are real.

After using the cloning transformation (1–2) on the quantum state (5) and tracing out the machine state vector, the two-qubit reduced density operator describing the two clones is given by

$$\rho_{ab}^{\text{out}} = \alpha^2(1-2\xi)|00\rangle\langle00| + \frac{\alpha\beta}{\sqrt{2}}(1-2\xi)|00\rangle\langle+| + \frac{\alpha\beta}{\sqrt{2}}(1-2\xi)|+\rangle\langle00|$$
$$+ 2\xi|+\rangle\langle+| + \frac{\alpha\beta}{\sqrt{2}}(1-2\xi)|+\rangle\langle11| + \frac{\alpha\beta}{\sqrt{2}}(1-2\xi)|11\rangle\langle+| + \beta^2(1-2\xi)|11\rangle\langle11|$$

(6)

where we have used the following notations:
$_x\langle Y_0|Y_0\rangle_x = {}_x\langle Y_1|Y_1\rangle_x = \xi, {}_x\langle Y_0|Q_1\rangle_x = {}_x\langle Q_0|Y_1\rangle_x = {}_x\langle Q_1|Y_0\rangle_x = {}_x\langle Y_1|Q_0\rangle_x = \frac{\eta}{2}, |+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Also, we have used the relation $\eta = 1 - 2\xi$ in equation (6) to make the distortion between the state ρ^{id} and the one-qubit reduced state ρ_a^{out} input state independent.

The necessary and sufficient condition for a state ρ to be inseparable is that at least one of the eigen values of the partially transposed operator defined as $\rho_{m\mu,n\nu}^T = \rho_{m\nu,n\mu}$ is negative [8, 7]. This is equivalent to the condition that at least one of the two determinants

$$W_3 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} \end{vmatrix} \quad \text{and} \quad W_4 = \begin{vmatrix} \rho_{00,00} & \rho_{01,00} & \rho_{00,10} & \rho_{01,10} \\ \rho_{00,01} & \rho_{01,01} & \rho_{00,11} & \rho_{01,11} \\ \rho_{10,00} & \rho_{11,00} & \rho_{10,10} & \rho_{11,10} \\ \rho_{10,01} & \rho_{11,01} & \rho_{10,11} & \rho_{11,11} \end{vmatrix}$$

is negative.

Now we investigate the inseparability of the two-qubit density operator ρ_{ab}^{out} for different intervals of the machine parameter ξ.

For the density matrix ρ_{ab}^{out}, we calculate the determinants W_3 and W_4, which are given by

$$W_3 = \frac{\alpha^2\xi(1-2\xi)}{2}[2\xi - \beta^2(1-2\xi)], \qquad W_4 = \frac{1}{2}[\alpha^2\beta^2\xi(1-2\xi)^2(6\xi-1) - 2\xi^4.] \quad (7)$$

Further, we note that the relation $\eta = 1 - 2\xi$ reduces the Schwarz inequality $\eta \leqslant 2(\xi - 2\xi^2)^{\frac{1}{2}}$ to the inequality $\frac{1}{6} \leqslant \xi \leqslant \frac{1}{2}$. So we consider the following cases:

(i) for $\xi = \frac{1}{6}$ and $\xi = \frac{1}{2}$, it is clear that $W_4 < 0$ for all α, β and hence the two-qubit density operator ρ_{ab}^{out} is inseparable for all α, β.

(ii) In the interval $\frac{1}{6} < \xi < \frac{1}{2}$, there exist sub-intervals of the machine parameter ξ for which the density operator ρ_{ab}^{out} is inseparable for some interval of α^2. Now we discuss two subcases below where the range of α^2 is given for which ρ_{ab}^{out} is inseparable.

(iia) In the interval $0 < \alpha^2 < \frac{1-4\xi}{1-2\xi}$, the determinant $W_3 < 0$ and hence ρ_{ab}^{out} is inseparable when the machine parameter lies in the interval $\frac{1}{6} < \xi < \frac{1}{4}$.

(iib) Moreover, in the interval $\frac{1-4\xi}{1-2\xi} < \alpha^2 < \frac{1}{2} - \frac{\sqrt{A^2 - 8A\xi^4}}{2A}$ where $A = \xi(6\xi-1)(1-2\xi)^2$, we find that the determinant $W_3 > 0$ but $W_4 < 0$ and hence in this case the state ρ_{ab}^{out} is inseparable when the machine parameter ξ lies in the range $\frac{3-\sqrt{5}}{4} < \xi < \frac{1}{4}$.

An arbitrary state of a two-qubit system can be represented as

$$\rho = \frac{1}{4}\left[I \otimes I + r \cdot \sigma \otimes I + I \otimes s \cdot \sigma + \sum_{i,j=1}^{3} \lambda_{ij}\sigma_i \otimes \sigma_j \right], \quad (8)$$

where ρ acts on the Hilbert space $H = H_1 \otimes H_2$, I stands for the identity operator, $\{\sigma_i\}_{i=1}^3$ are the standard Pauli matrices, r, s are vectors in R^3, $r \cdot \sigma = \sum_{i=1}^3 r_i\sigma_i$ and $s \cdot \sigma = \sum_{i=1}^3 s_i\sigma_i$. The coefficients $\lambda_{ij} = \text{Tr}(\rho\sigma_i \otimes \sigma_j)$ form a real 3×3 matrix which we shall denote by $C(\rho)$.

Further, it is known that the state which does not violate the Bell-CHSH inequality must satisfy $M(\rho) \leqslant 1$, where $M(\rho) = \max_{i>j}(u_i + u_j)$, u_i and u_j are the eigen values of $U = C^t(\rho)C(\rho)$ [11].

In our case, the elements of the correlation matrix $C(\rho_{ab}^{\text{out}})$ obtained for the density operator ρ_{ab}^{out} are

$$c_{11} = 2\xi, \qquad c_{12} = 0, \qquad c_{13} = 0, \qquad c_{21} = 0, \qquad c_{22} = -2\xi,$$
$$c_{23} = 0, \qquad c_{31} = 0, \qquad c_{32} = 0, \qquad c_{33} = 1 - 4\xi. \tag{9}$$

The eigen values of the matrix $U = C^t(\rho_{ab}^{\text{out}})C(\rho_{ab}^{\text{out}})$ are
$u_1 = u_2 = 4\xi^2$, $u_3 = 1 - 8\xi + 16\xi^2$.
Now we are in a position to discuss the different cases that establish a relation between the Bell-CHSH inequality violation and inseparability.

Case-I. If $u_1 > u_3$ then the machine parameter ξ lies between $\frac{1}{6}$ and $\frac{1}{2}$.

In $\frac{1}{6} < \xi < \frac{1}{2\sqrt{2}}$, $M(\rho) = 8\xi^2 < 1$ and hence in this interval the two-qubit density operator does not violate the Bell-CHSH inequality. However in the interval $\frac{1}{2\sqrt{2}} < \xi < \frac{1}{2}$, $M(\rho) = 8\xi^2 > 1$ and hence it violates the Bell-CHSH inequality.

Case-II. If $u_1 = u_2 = u_3 = \frac{1}{9}$ (which happens when $\xi = \frac{1}{6}$), then $M(\rho) < 1$, and hence the two-qubit entangled state does not violate the Bell-CHSH inequality.

Case-III. If $u_1 = u_2 = u_3 = 1$ (which happens when $\xi = \frac{1}{2}$), then $M(\rho) > 1$, and hence the two-qubit entangled state violates the Bell-CHSH inequality.

For any mixed spin-$\frac{1}{2}$ state ρ the teleportation fidelity [2] amounts to $F^{\text{max}} = \frac{1}{2}(1 + \frac{1}{3}N(\rho))$, where $N(\rho) = \sum_{i=1}^{3}\sqrt{u_i}$.

We split the interval $\frac{1}{6} \leqslant \xi < \frac{1}{2}$ into two sub-intervals $\frac{1}{6} \leqslant \xi \leqslant \frac{1}{4}$ and $\frac{1}{4} < \xi < \frac{1}{2}$. We notice that in the interval $\frac{1}{6} \leqslant \xi \leqslant \frac{1}{4}$, the teleportation fidelity does not cross the classical limit $\frac{2}{3}$. Hence for this interval we cannot use ρ_{ab}^{out} as a teleportation channel. However, in the interval $\frac{1}{4} < \xi < \frac{1}{2}$ the fidelity F^{max} crosses $\frac{2}{3}$. So for this interval of ξ, the output state ρ_{ab}^{out} can be used as a teleportation channel.

We note that in the case $\xi = \frac{1}{2}$, ρ_{ab}^{out} reduces to a maximally pure entangled state which can be used in teleportation with fidelity 1.

3. Conclusions

To summarize, we have cited an example of a two-qubit entangled state (output of the Buzek–Hillery cloning machine) which does not violate the Bell-CHSH inequality for some interval of the machine parameter ξ. Also, we have shown that the two-qubit entangled state does violate the Bell-CHSH inequality for some interval of the machine parameter ξ different from the previous one. In some of the cases, we found that the two-qubit entangled state does act as a useful quantum channel for the teleportation protocol. This work adds a special feature to the Buzek–Hillery quantum cloning machine.

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